Sum-free sets and related sets

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Abstract

A set A of integers is *sum-free* if $A \cap (A+A) = \emptyset$. Cameron conjectured that the number of sum-free sets $A \subseteq \{1, \ldots, N\}$ is $O(2^{N/2})$. As a step towards this conjecture, we prove that the number of sets $A \subseteq \{1, \ldots, N\}$ satisfying

$$(A+A+A) \cap (A+A+A+A) = \emptyset$$

is $2^{[(N+1)/2]}(1+o(1))$.

1 Introduction

We use the notation

$$\begin{aligned} A+B &= \{a+b : a \in A,\ b \in B\}, \\ A-B &= \{a-b : a \in A,\ b \in B\}, \\ A+x &= \{a+x : a \in A\}, \end{aligned}$$

etc., where $A, B \subseteq \mathbf{Z}$ and $x \in Z$.

A symbol "$O(\ldots)$" or "$\ll$" provided with an index implies a constant depending on the parameter(s) in the index. When there is no index, the implied constant is absolute.

A set $A \subset \mathbf{Z}$ is a *sum-free* if

$$A \cap (A+A) = \emptyset. \tag{SF}$$

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Cameron (see [4]) conjectured that the number $\mathrm{SF}(N)$ of sum-free sets $A \subseteq \{1, \ldots, N\}$ satisfies $\mathrm{SF}(N) \ll 2^{N/2}$. Note that the exponent $N/2$ cannot be improved because any set of odd numbers is sum-free, as well as any subset of $\{[N/2]+1, \ldots, N\}$.

The conjecture of Cameron is still open, in spite of a number of partial or similar results. For instance, Erdős and Granville (personal communication), Alon [1, Proposition 4.1], and Calkin [2] had independently proved that

$$\mathrm{SF}(N) = 2^{N/2+o(N)} \tag{1}$$

as $N \to \infty$. Cameron and Erdős [4] proved that there are at most $O\left(2^{N/2}\right)$ sum-free subsets of $\{[N/3], \ldots, N\}$.

Deshouillers, Freiman, Sós, and Temkin [5] investigated the structure of sum-free sets $A \subseteq \{1, \ldots, N\}$ with the additional condition

$$|A| \geq 2N/5 - x, \tag{2}$$

where x is a fixed positive integer. They obtained quite an explicit characterization, which, together with the result of Cameron and Erdős quoted above, yields that the number of sum-free sets $A \subseteq \{1, \ldots, N\}$ subject to (2) is $O_x\left(2^{N/2}\right)$.

On the other hand, as pointed to me J.-M. Deshouillers, Calkin's proof of (1) can be adapted to show that the number of sum-free sets $A \subseteq \{1, \ldots, N\}$ with at most $(1/4 - \delta)\, N$ elements is $O_\delta\left(2^{N/2}\right)$. Implementing this idea, we obtain the following result.

Theorem 1.1 *For any $\delta > 0$ there are at most $O_\delta\left(2^{\left(1/2-\delta^2/16\right)N}\right)$ sum-free sets $A \subseteq \{1, \ldots, N\}$ such that*

$$|A| \leq (1/4 - \delta)\, N. \tag{3}$$